MATHEMATICS

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AIM POINTS
MATHEMATICS
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XIth, XIIth, TARGET IIT-JEE
(MAIN + ADVANCE) & COMPETITIVE EXAM.
FOR XII (PQRS)

TANGENTS AND NORMALS

& Their Properties

CONTENTS

THINGS TO REMEMBER

1. If $y = f(x)$, then

$\left(\frac{dy}{dx}\right)_P$ = Slope of the tangent to $y = f(x)$ at point P.

$\frac{1}{\left(\frac{dy}{dx}\right)_P}$ = Slope of the normal to $y = f(x)$ at point P.

If the tangent is parallel to x-axis $\frac{dy}{dx} = 0$.

If the tangent is parallel to y-axis, then $\frac{dx}{dy} = 0$.

2. If $P(x_1, y_1)$ is a point on the curve $y = f(x)$, then

$y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$ is the equation of tangent at P.

$y - y_1 = \frac{1}{\left(\frac{dy}{dx}\right)_P}(x - x_1)$ is the equation of the normal at P.

3. The angle between the tangents to two given curves at their point of intersection is defined as the angle of intersection of two curves.

If C_1 and C_2 are two curves having equations $y = f(x)$ and $y = g(x)$ respectively such that they intersect at point P.

the angle θ of intersection of these two curves is given by

$$\tan\theta = \frac{\left(\frac{dy}{dx}\right)_{C_1} - \left(\frac{dy}{dx}\right)_{C_2}}{1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}}$$

If the angle of intersection of two curves is a right angle, then the curve are said to intersect orthogonally. The condition for orthogonality of two curves C_1 and C_2 is

$$\left(\frac{dy}{dx}\right)_{C_1} \times \left(\frac{dy}{dx}\right)_{C_2} = -1$$

4. Two curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ will intersect orthogonally, if

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$$

EXERCISE-1

1. Show that the tangents to the curve $y = 2x^3 - 3$ at the points where $x = 2$ and $x = -2$ are parallel.
2. Prove that the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) are at right angles.
3. Find the slope of the normal to the curve $x = 1 - a \sin \theta$, $y = b \cos^2 \theta$ at $\theta = \frac{\pi}{2}$
4. Find the slope of the normal to the curve $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ at $\theta = \frac{\pi}{4}$.
5. Find the point on the curve $y = 2x^2 - 6x - 4$ at which the tangent is parallel to the x-axis.
6. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.
7. Find the points on the curve $4x^2 + 9y^2 = 1$, where the tangents are perpendicular to the line $2y + x = 0$.
8. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent has the equation $y = x - 11$.
9. If the rangent to the curve $y = x^3 + ax + b$ at (1, -6) is parallel to the line $x - y + 5 = 0$, find a and b.
10. At what points on the circle $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangent is parallel to x-axis.
11. Find the points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangent are parallel to the (i) x-axis, (ii) y-axis.
12. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.
13. Show that the tangent to the curve $y = 7x^3 + 11$ at the points $x = 2$ and $x = -2$ are parallel.
14. Find the points on the curve $y = x^3$ where the slope of the tangents equal to the x-coordinate of the point.
15. Find the equations of the tangent and normal to the parabola $y^2 = 4a\,x$ at the point $(at^2, 2at)$.
16. Find the equation of the tangent line to the curve $x = 1 - \cos \theta$, $y = \theta - \sin \theta$ at $\theta = \frac{\pi}{4}$.
17. Find the equations of the tangent and the normal at the point 't' on the curve $x = a \sin^3 t$, $y = b \cos^3 t$.
18. Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where it crosses the y-axis.
19. Find the equations of the tangent and the normal to the curve $y = \frac{x-7}{(x-2)(x-3)}$ at the point, where it cuts x-axis.
20. Find the equation of the tangent to the curve $y = (x^3 - 1)(x - 2)$ at the points where the curve cuts the x-axis.
21. Find the equation of tangent to the curve $y = x^3 + 2x + 6$ which is perpendicular to the line $x + 14y + 4 = 0$.
22. Find all the tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$ that are parallel to the line $x + 2y = 0$.
23. For the curve $y = 4x^3 - 2x^5$ find all points at which the tangent passes through the origin.
24. Find the equations of all lines of slope -1 that are tangents to the curve $y = \frac{1}{1-x}$, $x \neq 1$.

25. Prove that all normals to the curve $x = a\cos t + at\sin t$, $y = a\sin t - at\cos t$ are at a distance a form the origin.
26. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2).
27. Find the equations of the tangent and the normal to the following curves at the indicated points :
    (i) $y = x^4 - bx^3 + 13x^2 - 10x + 5$ at (0, 5)
    (ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)
    (iii) $y = x^2$ at (0, 0)
    (iv) $y = 2x^3 - 3x - 1$ at (1, -2)
    (v) $y^2 = \frac{x^3}{4-x}$ at (2, -2)
    (vi) $y = x^2 + 4x + 1$ at $x = 3$
    (vii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$
    (viii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec\theta, b\tan\theta)$
    (ix) $y^2 = 4ax$ at $(a/m^2, 2a/m)$
    (x) $c^2(x^2 + y^2) = x^2y^2$ at $\left(\frac{c}{\cos\theta}, \frac{c}{\sin\theta}\right)$
    (xi) $xy = c^2$ at$(ct, c/t)$
    (xii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1)
    (xiii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_0, y_0)
    (xiv) $x^{2/3} + y^{2/3} = 2$ at (1, 1)
    (xv) $x^2 = 4y$ at (2, 1)
    (xvi) $y^2 = 4x$ at (1, 2)
28. Find the equation of the normal to the curve $y^2 = x^3$ at the point (am^2, am^3).
29. Find the eqation of the tangent line to the curve $y = x^2 - 2x + 7$ which is
    (i) parallel to the line $2x - y + 9 = 0$
    (ii) perpendicular to the line $5y - 15x = 13$
30. Find the equations of all lines of slope zero and that are tangent to the curve $y = \frac{1}{x^2 - 2x + 3}$.

31. Find the equation of the tangent to the curve $x^2 + 3y - 3 = 0$, which is parallel to the line $y = 4x - 5$.

32. Find the equation of the tangent to the curve $x = \sin 3t$, $y = \cos 2t$ at $t = \frac{\pi}{4}$.

33. Show that the condition that the curves

$$ax^2 + by = 1 \quad ...(i)$$

and, $$a' x^2 + b' y^2 = 2 \quad ...(ii)$$

shold intersect orthogonally is that $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$.

34. Show that the curves $x = y^2$ and $xy = k$ cut at right angles, if $8k^2 = 1$.

35. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

36. Show that the angle between the tangent at any point P and the line joining P to the origin O is the same at all points on the curve

$$\log(x^2 + y^2) = k \tan^{-1}\left(\frac{y}{x}\right)$$

37. Find the angle of intersection of the following cuves :
   (i) $y^2 = x$ and $x^2 = y$
   (ii) $y = x^2$ and x^2 and $x^2 + y^2 = 20$
   (iii) $2y^2 = x^3$ and $y^2 = 32x$
   (iv) $x^2 + y^2 - 4x - 1 = 0$ and $x^2 + y^2 - 2y - 9 = 0$
   (v) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x^2 + y^2 = ab$
   (vi) $x^2 + 4y^2 = 8$ and $x^2 - 2y^2 = 2$
   (vii) $x^2 = 27y$ and $y^2 = 8x$
   (viii) $x^2 + y^2 = 2x$ and $y^2 = x$

38. Find the condition for the following set of curves to intersect orthogonally :
   (i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $xy = c^2$
   (ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1 = 1$

39. Show that the curve $\frac{x^2}{a^2+\lambda_1} + \frac{y^2}{b^2+\lambda_1} = 1$ and $\frac{x^2}{a^2+\lambda_2} + \frac{y^2}{b^2+\lambda_2} = 1$ intersect at right angles.

EXERCISE-2

1. Write the angle made by the tangent to the curve $x = e^t \cot t$, $y = e^t \sin t$ at $t = \frac{\pi}{4}$ with the x-axis.
2. Write the equation of the tangent to the curve $y = x^2 - x + 2$ at the point where it crosses the y-axis.
3. Write the coordinates of the point at which the tangent to the curve $y = 2x^2 - x + 1$ is parallel to the line $y = 3x + 9$.

EXERCISE-3

1. The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t = 5$ at point (2, -1) is

(a) $\frac{22}{7}$ (b) $\frac{6}{7}$ (c) -6 (d) none of these